January 2022

Shift Operator E

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y,

then $E y_r = y_{r+h}$, $E^2 y_r = y_{r+2h}$

$$E^m y_r = y_{r+mh}$$

Representation

$$\Delta y_0 = y_1 - y_0 = E y_0 - y_0 = (E-1) y_0$$

$(E - 1)(y_0)$

So, $\Delta = E - 1$

or, $E = \Delta + 1$

Ex. Find $\Delta^4 y_0$ using E operator.

Ans. $\Delta^4 y_0 = (E-1)^4 y_0$

$$= (E^4 - 4E^3 + (E^2 - 4E + 1) y_0$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

V.V. Imp

H.W. Prove that:

$$\nabla = 1 - E^{-1}, \quad \delta = (E^{1/2} - E^{-1/2})$$

$$\Delta \equiv \nabla E \equiv \delta E^{1/2}$$

H.W. Show that,

$$f(x_0 + nh) = f(x_0) + \binom{n}{1} \Delta f(x_0) + \binom{n}{2} \Delta^2 f(x_0) + \cdots + \binom{n}{n} \Delta^n f(x_0)$$

H.W. Use method of "separation of symbol" to show
(Solution done)

(1) $$\Delta^n u_{x-n} = u_x - n\, u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

Solution

$$\text{RHS} = u_x - nE^{-1}u_x + \frac{n(n-1)}{2}E^{-2}u_x + \cdots + (-1)^n E^{-n}u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + \cdots + (-1)^n E^{-n}\right]u_x$$

$$= (1 - E^{-1})^n u_x$$

$$= \frac{\Delta^n}{E^n}u_x = \Delta^n E^{-n} u_x$$

$$= \Delta^n u_{x-n}$$

$$= \text{LHS}$$

H.W. Use method of separation to show:

(1) $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \dots\dots\infty$

$e^x = 1 + x + \frac{x^2}{2!} + \dots$

$= e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \dots\infty\right)$

(2) $u_0 - u_1 + u_2 - u_3 + \dots$

$= \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \dots$

(3) $$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \dots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-(n+1)}$$

(4) $$u_1 + u_2 + u_3 + \cdots + u_n = \binom{n}{1} u_1 + \binom{n}{2} \Delta u_1 + \cdots + \Delta^{n-1} u_1$$

Differences of zero

The term $\Delta^n x^m \Big|_{x=0}$ is known as "differences of zero" and denoted by

$$\Delta^n 0^m$$ where n, m are both positive integers.

Ex: Show that,

$$\Delta^n 0^m = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \dots + (-1)^n 0^m$$

Ans: $\Delta^n x^m = (E-1)^n x^m$

$$= \left[E^n - \binom{n}{1}E^{n-1} + \binom{n}{2}E^{n-2} - \dots + (-1)^n\right] x^m$$

$$= (x+n)^m - \binom{n}{1}(x+n-1)^m + \binom{n}{2}(x+n-2)^m - \dots + (-1)^n x^m$$

Put $x=0$ to get

$$\Delta^n 0^m = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \dots + (-1)^n 0^m$$

H.W. If $f(x)$ is a polynomial in x of degree m, then prove that

$$x^m = \sum_{i=0}^{m} \binom{x}{i} \Delta^i 0^m$$

☆

Try yourself. I am leaving it blank so that I can write down the solution, if needed.

Jan 2022

Newton's Interpolation

(Newton's Forward interpolation formula)

Suppose we have $(n+1)$ "equispaced" values of x, say $x_0, x_1, x_2, \cdots, x_n$ where $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$ and corresponding function values $f(x)$ are $y_0, y_1, \cdots, y_n$.

Our objective is to find a polynomial $\phi_n(x)$ of degree n, which replaces $f(x)$

on the given set of points.

Assuming $f(x)$ to be continuous in (x_0, x_n), we may replace $f(x)$ by a polynomial $\phi_n(x)$ of degree n in x such that

$$f(x_i) = \phi_n(x_i) \qquad i = 0, 1, 2, \cdots, n$$

But for all other points

$$f(x) = \phi_n(x) + R_n(x)$$

where $R_n(x)$ is the error term.

Ignoring $R(x)$ for now, we may write

$$f(x) \approx \phi_n(x)$$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1})$$

put $x = x_0$, we have $a_0 = f(x_0) = y_0$

put $x = x_1$, we have

$$y_1 = y_0 + a_1 h \Rightarrow a_1 = \frac{\Delta y_0}{h}$$

put $x = x_2$, we have

$$y_2 = y_0 + \frac{\Delta y_0}{h} \cdot 2h + a_2 \cdot 2h \cdot h$$

$$\Rightarrow 2h^2 a_2 = y_2 - 2y_1 + y_0 \Rightarrow a_2 = \frac{\Delta^2 y_0}{2!\, h^2}$$

proceeding this way,

$$a_n = \frac{\Delta^n y_0}{n!\, h^n}$$

Substituting the values of $a_0, a_1, a_2, \cdots, a_n$ in previous eq; we have

$$f(x) \simeq y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^n y_0}{n!\, h^n}(x - x_0) \cdots (x - x_{n-1})$$

Let $\frac{x - x_0}{h} = u$

(This is a dimensionless quantity called phase)

$x - x_1 = x - x_0 + x_0 - x_1 = uh - h = (u-1)h$

$x - x_2 = x - x_1 + x_1 - x_2 = (u-1)h - h = (u-2)h$

so on

$$\underbrace{f(x) \simeq y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n y_0}$$

This expression is known as Newton's Forward interpolation formula

* Forward interpolation formula is useful for interpolation near the beginning of a set of tabular values.

Newton's Backward Interpolation Formula

Suppose we have $(n+1)$ equispaced values of x, say $x_0, x_1, x_2, \dots, x_n$ where $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, n$ and corresponding function values $f(x)$ are $y_0, y_1, \dots, y_n$.

Now our objective is to find a polynomial $\phi_n(x)$ of degree n, which replaces $f(x)$ on the given set of points.

Assuming $f(x)$ to be continuous in (x_0, x_n).
we may replace $f(x)$ by a polynomial $\phi_n(x)$ of deg n in x s.t

$$f(x_i) = \phi_n(x_i) \qquad i = 0, 1, 2, \cdots, n$$

But for all other points

$$f(x) = \phi_n(x) + R_n(x) \qquad [\text{Remainder term}]$$

$$f(x) \approx \phi_n(x)$$

$$= a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots + \ldots + a_n(x - x_n)\cdots(x - x_1)$$

putting $x = x_n$ we have $a_0 = f(x_n) = y_n$

put $x = x_{n-1}$, $y_{n-1} = y_n + a_1(-h)$

$$\Rightarrow a_1 = \frac{\nabla y_n}{h}$$

put $x = x_{n-2}$, $y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2(-2h)(-h)$

$$\Rightarrow \quad 2h^2 a_2 = y_n - 2y_{n-1} + y_{n-2}$$

$$\Rightarrow \quad a_2 = \frac{\nabla^2 y_n}{2!\, h^2}$$

proceeding this way, $a_n = \dfrac{\nabla^n y_n}{n!\, h^n}$

Substituting the values $a_0, a_1, a_2, \ldots\ldots,$ we have

$$f(x) \approx y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!\, h^2}(x - x_n)(x - x_{n-1})$$

$$+ \dots + \frac{\nabla^n y_n}{n!\, h^n}(x - x_n) \dots (x - x_1)$$

Let $\frac{x - x_n}{h} = u$

$x - x_{n-1} = x - x_n + x_n - x_{n-1} = uh + h = (u+1)h$

$x - x_{n-2} = (u+2)h$ so on

$$\Rightarrow \quad f(x) \simeq y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots + \frac{u(u+1)\cdots(u+n-1)}{n!}\nabla^n y_n$$

This expression is known as Newton's Backward interpolation formula.

It is useful for interpolation near the end of a set of tabular values

https://nptel.ac.in/content/storage2/courses/122104018/node109.html